Consistency of Orbifold Conformal Field Theories on $K3$

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Abstract

We explicitly determine the locations of G orbifold conformal field theories, $G=\mathbb{Z}_M$, $M\in\{2,3,4,6\}$, $G=\widehat{D}_n$, $n\in\{4,5\}$, or G the binary tetrahedral group $\widehat{\mathbb{T}}$, within the moduli space $\mathcal{M}^{K3}$ of $N=(4,4)$ superconformal field theories associated to $K3$. This is achieved purely from the known description of the moduli space [AM94] and the requirement of a consistent embedding of orbifold conformal field theories within $\mathcal{M}^{K3}$. We calculate the Kummer type lattices for all these orbifold limits. Our method allows an elementary derivation of the B–field values in direction of the exceptional divisors that arise from the orbifold procedure [Asp95, Dou97, BI97], without recourse to D–geometry. We show that our consistency requirement fixes these values uniquely and determine them explicitly. The relation of our results to the classical McKay correspondence is discussed.

1 Introduction

In this paper, we study certain subvarieties of the moduli space $\mathcal{M}$ of $N=(4,4)$ superconformal field theories with central charge $c=6$. More precisely, all theories in $\mathcal{M}$ are assumed to be representations of the $N=(4,4)$ linear extension of the $N=(2,2)$ superconformal algebra that contains $su(2)_l\oplus$

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$su(2)_r$, a special case of the Ademollo et al. algebra [ABD$^+$76]. Moreover, with respect to a Cartan subalgebra of $su(2)_l \oplus su(2)_r$, left and right charges (i.e. doubled spins) of each state in our superconformal field theories are assumed to be integral. The structure of $\mathcal{M}$ has already been described in detail [AM94, Asp97, RW98, Dij99, NW01]. Let us summarize its most important features.

$\mathcal{M}$ decomposes into two components, $\mathcal{M} = \mathcal{M}^{tori} \cup \mathcal{M}^{K3}$. Every theory in $\mathcal{M}$ can be assigned to either the torus or the $K3$ component of the moduli space by its elliptic genus, which vanishes in the torus case and reproduces the geometric elliptic genus of $K3$ otherwise [EOTY89, NW01]. Each irreducible component of $\mathcal{M}$ is locally described by a Grassmannian $\mathcal{T}^{4,4+\delta}$ [Nar86, Sei88, Cec91] (see Appendix B for notations and properties of Grassmannians). Here, $\delta = 0$ for the torus component and $\delta = 16$ for $K3$. Hence the defining data of a superconformal field theory in $\mathcal{M}$ have been encoded by a positive definite four-plane $x \subset \mathbb{R}^{4,4+\delta}$. Provisionally, let X denote a complex two-torus or a $K3$ surface, depending on which component of the moduli space x belongs to. Then $\mathbb{R}^{4,4+\delta} \cong H^{even}(X,\mathbb{R})$, where on cohomology we always use the scalar product which is induced by the intersection pairing on X. The four-plane x is now interpreted as subspace of $H^{even}(X,\mathbb{R})$. By Poincaré duality, $H^{even}(X,\mathbb{Z})$ is an even self-dual lattice of signature $(4, 4+\delta)$ (see Appendix A for some mathematical background on lattices). Hence by Theorem A.1, $H^{even}(X,\mathbb{Z})$ is uniquely determined up to lattice automorphisms, and we assume that an embedding $H^{even}(X,\mathbb{Z}) \hookrightarrow H^{even}(X,\mathbb{R})$ has been chosen. Then the four-plane $x \subset H^{even}(X,\mathbb{R})$ is specified by its relative position with respect to $H^{even}(X,\mathbb{Z})$.

Each theory in $\mathcal{M}^{tori}$ has a description as nonlinear sigma model with target space a complex two-torus. The moduli space of toroidal conformal field theories had originally been given by Narain [Nar86] in terms of the odd torus cohomology. To arrive at the above description in terms of the even torus cohomology one has to use $SO(4,4)$ triality, see [NW01].

For $\mathcal{M}^{K3}$ one uses the isomorphism (B.1) with primitive null vectors $\upsilon, \upsilon^0 \in H^{even}(X,\mathbb{Z})$, $\langle \upsilon, \upsilon^0 \rangle = 1$, to show that its parameter space agrees with the parameter space of nonlinear sigma models with $K3$ target [AM94]. Here, υ, υ^0 are interpreted as generators of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$, respectively. This description equally holds in the torus case [NW01]. The image (Σ, V, B) of a given four-plane x under (B.1) is called a *geometric interpretation*. Here, the three-plane $\Sigma \subset H^2(X,\mathbb{R}) \cong \mathbb{R}^{3,3+\delta}$ is interpreted as the subspace of self-dual two–forms and thus encodes an Einstein metric of volume 1 on X. The three-plane Σ is specified by its relative position with respect to the even self-dual lattice $H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$ (see Theorem

A.1). The parameter V is interpreted as volume of X, and $B \in H^2(X,\mathbb{R})$ denotes the B-field. We remark that in contrast to higher dimensional Calabi Yau manifolds we need not perform a large volume limit in order to study nonlinear sigma models on $K3$, since here the metric on the moduli space does not receive instanton corrections [NS95].

Globally, the irreducible components of $\mathcal{M}$ are obtained by modding out a discrete symmetry group from their local descriptions. Namely,

$$\mathcal{M}^\delta = O^+(H^{even}(X,\mathbb{Z}))\backslash O^+(H^{even}(X,\mathbb{R}))/SO(4)\times O(4+\delta) \qquad (1.1)$$

[Nar86, AM94], up to a subtlety that results in the choice of $O^+(H^{even}(X,\mathbb{Z}))$ instead of $O(H^{even}(X,\mathbb{Z}))$ above [NW01]. In order to generate the group $O^+(H^{even}(X,\mathbb{Z}))$ one firstly needs the *classical symmetries* which identify equivalent Einstein metrics thus fixing υ, υ^0. Secondly, we have B-field shifts by $\lambda \in H^2(X,\mathbb{Z})$ which induce $(\upsilon,\upsilon^0) \mapsto (\upsilon, \upsilon^0 + \lambda - \frac{\lambda^2}{2}\upsilon)$ and for w with $\langle w,\upsilon\rangle = 0$ induce $w \mapsto w - \langle\lambda, w\rangle\upsilon$. Thirdly, one can use mirror symmetry [AM94, AM], or the Fourier–Mukai transform $\upsilon \leftrightarrow \upsilon^0$ [NW01], where the latter approach appears to be the simpler one.

Given the above description of $\mathcal{M}$, we formulate the aim of this paper as follows: Consider a superconformal field theory in $\mathcal{M}^{tori}$, specified by a four-plane $x_T \in \mathcal{T}^{4,4}$, that admits a discrete symmetry G which preserves supersymmetry, so $G \subset SU(2)$. Then for nontrivial non-translational G the resulting G orbifold conformal field theory is known to belong to $\mathcal{M}^{K3}$ (see, e.g., [EOTY89] to check the elliptic genera in the case of cyclic groups $G = \mathbb{Z}_M$, $M \in \{2,3,4,6\}$). For all possible such actions that do not contain non-trivial translations (by [Fuj88] this means for $G = \mathbb{Z}_M$, $M \in \{2,3,4,6\}$, $G = \widehat{D}_n$, $n \in \{4,5\}$, and $G = \widehat{\mathbb{T}}$), we specify the location of the resulting four-plane $x \in \mathcal{T}^{4,20}$ in a way that allows to explicitly read off a geometric interpretation of x on the corresponding G orbifold limit of $K3$. In particular, we show that a consistent embedding of the subvarieties which contain such orbifold conformal field theories in $\mathcal{M}^{K3}$ fixes the B-field values of the above geometric interpretation in direction of the exceptional divisors of the blow up of the orbifold singularities. We determine these B-field values explicitly.

As a first step, in Sect. 2, we describe the underlying geometric picture. In other words, we specify the locations of orbifold limits within the moduli space $O^+(H^2(X,\mathbb{Z}))\backslash\mathcal{T}^{3,19}$ of Einstein metrics with volume 1 on $K3$. In contrast to the non-compact minimal resolution of $\mathbb{C}^2/G$, the components of the exceptional divisors on X do not generate a primitive sublattice of $H_2(X,\mathbb{Z})$. For $G = \mathbb{Z}_2$, in [PŠŠ71, Nik75] it was shown that they are rather contained in a finer lattice, the *Kummer lattice*. We explicitly calculate the generalizations of the Kummer lattice to all G listed above. In Sect. 3 we use

Theorems A.2 and B.1 to lift these geometric results to the "quantum level". This in particular leads to a short derivation of the correct B–field values for orbifold conformal field theories which is essentially independent of the technical discussion in Sect. 2. We are in agreement with previous results by Aspinwall [Asp95], Douglas [Dou97] and Blum/Intriligator [BI97]. We conclude with a summary and discussion of possible further implications our techniques might have. We briefly point out their surprisingly simple relation to the classical McKay correspondence. There are two appendices to present the necessary mathematical background on lattices and Grassmannians.

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2 Kummer type constructions of $K3$

Apart from notations which are introduced at the beginning of the present section, further results of this work can be understood without the technical details discussed below. In particular, the proofs in Sect. 3 are mostly independent of Prop. 2.1.

Let T denote a complex two–torus with Einstein metric of volume 1 specified by the positive definite three–plane $\Sigma \subset H^2(T,\mathbb{R})$ of self dual two–forms. Assume that T possesses a nontrivial discrete symmetry $G \subset SU(2)$, such that the induced action on Σ is trivial. The variety T/G has a set $\mathcal{S}$ of singularities of ADE type [Val34, Art66], which we assume to be nonempty. By $\widetilde{\mathcal{S}} \subset T$ we denote the pre-image of $\mathcal{S}$ in T. The minimal resolution $p: X \longrightarrow T/G$ produces a $K3$ surface X *in the orbifold limit*. This means that on X we use the metric which is induced by the flat torus metric and assigns volume zero to all components of exceptional divisors $p^{-1}(s), s \in \mathcal{S}$.

Here, we restrict considerations to groups G that do not contain nontrivial translations. In [Fuj88], such G actions have been classified, and our discussion below covers all these cases (see (2.2)). In fact, this means that we omit only two further orbifold constructions of $K3$ where G does contain translations [Mor], [dBDH$^+$, Table 18]. To locate all such G orbifold limits within the moduli space of volume 1 Einstein metrics on $K3$, we will determine the appropriate embedding of the image of G invariant torus forms $H^2(T,\mathbb{Z})^G$ in $H^2(X,\mathbb{Z})$. Since the geometric data of Einstein metrics

on T, X are given by three–planes in $H^2(T,\mathbb{R})$, $H^2(X,\mathbb{R})$ which are specified by their relative position with respect to these lattices, such an embedding indeed is all we need to find.

As to notations, for $s\in\mathcal{S}$ the irreducible components of the exceptional divisor $p^{-1}(s)$ are rational spheres with intersection matrix the negative of the Cartan matrix corresponding to the type of singularity s. Their Poincaré duals are lattice vectors in $H^2(X,\mathbb{Z})$, which in this section we denote* $E_s^{(j)}, (E_s^{(j)})^2=-2$, and which span an ADE type root lattice Γ_s. We set $\mathcal{E}_{|G|} := \bigcup_{s\in\mathcal{S}} \{E_s^{(j)}\}$ and $\Gamma_{|G|} := \bigoplus_{s\in\mathcal{S}} \Gamma_s \subset H^2(X,\mathbb{Z})$. Note that $\Gamma_{|G|}$ is a root lattice with fundamental system $\mathcal{E}_{|G|}$.

On T, we choose complex coordinates z_1,z_2 compatible with the Einstein metric and split T into $T=T^2\times\widetilde{T}^2$ with elliptic curves T^2, $\widetilde{T}^2$. Both curves are assumed to be $\mathbb{Z}_M$ symmetric, but the metric need not be diagonal with respect to z_1,z_2. If $M\in\{2,3,4,6\}$, we consider the algebraic $G=\mathbb{Z}_M$ action, where $\mathbb{Z}_M$ is realized as group of M^{th} roots of unity in $\mathbb{C}$:

$$(z_1,z_2)\in\mathbb{C}^2,\ \mathbb{Z}_M\ni\zeta: \qquad \zeta.(z_1,z_2)=(\zeta z_1,\overline{\zeta}z_2);$$

if $T^2=\widetilde{T}^2$ and $M\in\{4,6\}$ we also have algebraic $\widehat{D}_{M/2+2}$ actions with additional generator I,

$$(z_1,z_2)\in\mathbb{C}^2: \quad I.(z_1,z_2)=(-z_2,z_1).$$

We have to rewrite the results of [Fuj88, Table 9] to notice that the above $\widehat{D}_4$ action is algebraic as well on tori

$$\begin{aligned} T_{\mathbb{T}} &:= \mathbb{C}^2/\Lambda_{D_4}, \\ \Lambda_{D_4} &:= V\,\mathrm{span}_{\mathbb{Z}}\left\{(1,0),\,(i,0),\,(\tfrac{i+1}{2},\tfrac{i+1}{2}),\,(\tfrac{i+1}{2},\tfrac{i-1}{2})\right\}, \quad V\in\mathbb{R}, \end{aligned} \tag{2.1}$$

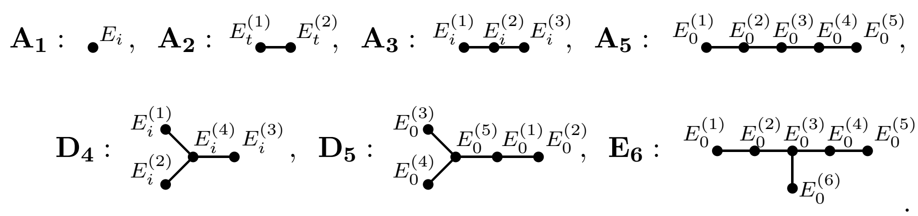
but with different fixed point sets (see (2.2)). To distinguish the two cases, the latter is denoted $\widehat{D}_4'$. On $T_{\mathbb{T}}$ as in (2.1) there also is an algebraic action of the binary tetrahedral group $\widehat{\mathbb{T}}$ which is obtained from the $\widehat{D}_4'$ action with additional generator

$$(z_1,z_2)\in\mathbb{C}^2: \quad J.(z_1,z_2)=\tfrac{i+1}{2}\,(\,i(z_1-z_2)\,,-(z_1+z_2)\,)\,.$$

From the torus geometry it is natural to label $\mathbb{Z}_2$ and $\mathbb{Z}_4$ type fixed points by vectors $i\in\mathbb{F}_2^4$, whereas $\mathbb{Z}_3$ type fixed points carry labels $t\in\mathbb{F}_3^2$ ($\mathbb{F}_p$, p

*This is a slight abuse of notation as we will see in Sect. 3, where the $E_s^{(j)}$ are replaced by $\widehat{E}_s^{(j)}$.

prime, denotes the unique finite field with p elements). The integral two–forms dual to the components of exceptional divisors in the blow up of an ADE singularity are labeled as follows:



Recall that a $\mathbb{Z}_m$ type fixed point gives an A_{m-1} type singularity on T/G, whereas $\widehat{D}_n$ type and $\widehat{\mathbb{T}}$ type fixed points correspond to D_n and E_6 type singularities, respectively. Then for the various orbifold limits of $K3$ we have

G	$\Gamma_{\lvert G\rvert}(-1)$	$\mathcal{E}_{\lvert G\rvert}$
$\mathbb{Z}_2$	A_1^{16}	$E_i,\ \ i\in\mathbb{F}_2^4,$
$\mathbb{Z}_3$	A_2^9	$E_t^{(l)}, t\in\mathbb{F}_3^2,\ l\in\{1,2\},$
$\mathbb{Z}_4$	$A_3^4\oplus A_1^6$	$E_i^{(l)}, i\in I^{(4)} := \{(j,k),\ j,k\in\{(0,0),(1,1)\}\}\,,$ $l\in\{1,2,3\},$ $E_i,\ \ i\in I^{(2)} := \left\{(j,1,0),(1,0,j),\ j\in\mathbb{F}_2^2\right\},$ $(1,0,1,0)\sim(0,1,0,1),$
$\mathbb{Z}_6$	$A_5\oplus A_2^4\oplus A_1^5$	$E_0^{(l)},\ l\in\{1,\dots,5\},$ $E_t^{(l)}, t\in\{(1,0),(0,1),(1,1),(1,2)\}\,,\ l\in\{1,2\},$ $E_i,\ \ i\in\{(1,0,0,0),(0,0,0,1),$ $(1,0,1,0),(1,0,0,1),(0,1,0,1)\}$
$\widehat{D}_4$	$D_4^2\oplus A_3^3\oplus A_1^2$	$E_i^{(l)}, i\in\{0,1\}\,, l\in\{1,\dots,4\}\,,$ $E_i^{(l)}, i\in\{(1,1,0,0),(1,0,1,0),(1,0,0,1)\}\,,$ $l\in\{1,2,3\},$ $E_i,\ \ i\in\{(1,0,0,0),(0,1,1,1)\}\,,$
$\widehat{D}'_4$	$D_4^4\oplus A_1^3$	$E_i^{(l)}, i\in I^{(4)} := \{(j,k), j,k\in\{(0,0),(1,1)\}\}\,,$ $l\in\{1,\dots,4\},$ $E_i,\ \ i\in\{(1,0,0,0),(0,0,0,1),(1,0,1,0)\},$

(2.2)

G	$\Gamma_{\lvert G\rvert}(-1)$	$\mathcal{E}_{\lvert G\rvert}$
$\widehat{D}_5$	$D_5 \oplus A_3^3 \oplus A_2^2 \oplus A_1$	$E_0^{(l)}, l \in \{1,\dots,5\}$, $E_i^{(l)}, i \in \{(1,0,1,0),(1,0,0,1),(0,1,0,1)\}$, $l \in \{1,2,3\}$, $E_t^{(l)}, t \in \{(1,0),(1,1)\}$, $l \in \{1,2\}$, $E_i,\ \ i=(0,0,0,1)$.
$\widehat{\mathbb{T}}$	$E_6 \oplus D_4 \oplus A_2^4 \oplus A_1$	$E_0^{(l)},\qquad l \in \{1,\dots,6\}$, $E_{(1,1,0,0)}^{(l)},\quad l \in \{1,\dots,4\}$, $E_t^{(l)},\ t \in \{(1,0),(0,1),(1,1),(1,2)\}$, $l \in \{1,2\}$, $E_{(1,0,0,0)}$.

(2.2)

From the very definition of the orbifold construction we have a rational map $\pi : T \longrightarrow X$ of degree $|G|$, which is defined outside the fixed points of G. By $K_{|G|}$, $\Pi_{|G|}$ we denote the primitive sublattices of $H^2(X,\mathbb{Z})$ that contain $\pi_*(H^2(T,\mathbb{Z})^G)$, $\Gamma_{|G|}$, respectively. From (2.2) one checks that rk $K_{|G|}+$ rk $\Pi_{|G|} = 22 =$ rk $H^2(X,\mathbb{Z})$, confirming that X indeed is a $K3$ surface. $K_{|G|} \perp \Pi_{|G|}$ by construction, since all exceptional divisors have volume zero with respect to any Hermitean metric compatible with the Einstein metric in the orbifold limit. Hence by Theorem A.2 there is an isomorphism $\gamma : (K_{|G|})^*/K_{|G|} \rightarrow (\Pi_{|G|})^*/\Pi_{|G|}$. Moreover, the embedding $\pi_*(H^2(T,\mathbb{Z})^G) \hookrightarrow H^2(X,\mathbb{Z})$ is determined, once we know the lattices $K_{|G|}$, $\Pi_{|G|}$, and γ. The rest of this section therefore is devoted to a geometrically motivated construction of these data.

Note that $\pi_*(H^2(T,\mathbb{Z})^G) \cong H^2(T,\mathbb{Z})^G(|G|)$ by [Ino76, Prop. 1.1]. Since we prefer to work with metric isomorphisms, we denote the π_* image of $\kappa \in H^2(T,\mathbb{Z})^G$ by $\sqrt{|G|}\kappa$. This is also in accord with the fact that $\pi_*\pi^* = |G|$ and $\pi^*\pi_* = |G|$ by [Ino76, Prop. 1.1].

Let us recall Nikulin's solution to our problem in the case $G = \mathbb{Z}_2$, i.e. for classical Kummer surfaces. In [PŠŠ71, Nik75], it is proven that $K_2 \cong H^2(T,\mathbb{Z})(2)$, and $\Pi := \Pi_2$, the *Kummer lattice*, is determined to

$$\Pi = \mathrm{span}_{\mathbb{Z}}\{E_i,\, i \in \mathbb{F}_2^4; \quad \tfrac{1}{2}\sum_{i\in H} E_i,\, H \subset \mathbb{F}_2^4 \text{ a hyperplane}\}. \tag{2.3}$$

We interpret the description of $H^2(X,\mathbb{Z})$ that arises from Theorem A.2 as follows: Clearly, $H^2(X,\mathbb{Z})$ contains Π and K_2. The latter consists of the Poincaré duals $\sqrt{2}\kappa$ of images of torus two–cycles that correspond to $\kappa \in$

$H^2(T,\mathbb{Z})$. These cycles must be in general position, i.e. must not meet $\mathbb{Z}_2$ fixed points, for κ to have a well defined image in $H^2(X,\mathbb{Z})$. Suppose the cycle does meet fixed points with labels in $P \subset \mathbb{F}_2^4$. Then the $\mathbb{Z}_2$ quotient produces a 2:1 cover of a sphere with branch points $s \in P$, which on blowing up are replaced by the corresponding exceptional divisors. Hence $\sqrt{2}\kappa - \sum_{s\in P} E_s$ is Poincaré dual to a 2:1 unbranched covering of a $K3$ cycle, i.e. $\frac{1}{\sqrt{2}}\kappa - \frac{1}{2}\sum_{s\in P} E_s \in H^2(X,\mathbb{Z})$. Indeed, the above describes a well defined map

$$\gamma: K_2^*/K_2 \longrightarrow \Pi^*/\Pi, \quad \gamma(\tfrac{1}{\sqrt{2}}\kappa) := -\tfrac{1}{2}\sum_{s\in P} E_s,$$

since for P, P' corresponding to two different non-generic positions of our cycle, $\frac{1}{2}\sum_{s\in P} E_s - \frac{1}{2}\sum_{s\in P'} E_s \in \Pi$ by (2.3).

Vice versa, Theorem A.2 together with (2.3) shows that $H^2(X,\mathbb{Z})$ is generated by

i. $\pi_*(H^2(T,\mathbb{Z})) \cong H^2(T,\mathbb{Z})(2)$,

ii. $\mathcal{E}_2$, the Poincaré duals of the rational spheres comprising the exceptional divisors in the blow up,

iii. forms of type $\frac{1}{\sqrt{2}}\kappa - \frac{1}{2}\sum_{s\in P} E_s \in H^2(X,\mathbb{Z})$, where $\sqrt{2}\kappa \in \pi_* H^2(T,\mathbb{Z})$ determines P as explained above.

In particular, the entire lattice $H^2(X,\mathbb{Z})$ is given in terms of two–forms that correspond to torus cycles or exceptional divisors, so for $G=\mathbb{Z}_2$ the desired embedding $\pi_*(H^2(T,\mathbb{Z})^G) \hookrightarrow H^2(X,\mathbb{Z})$ is found.

It is obvious how to generalize i., ii. above to the other groups G. To understand forms of type iii. consider the following calculation in terms of local coordinates for $G=\mathbb{Z}_M$: We choose $\mathbb{Z}_M$ invariant polynomials $(x_1,x_2,x_3) := (z_1^M, z_2^M, z_1 z_2)$ as coordinates on $T/\mathbb{Z}_M$ near the fixed point $(z_1,z_2)=(0,0)$. The blow up of $(0,0)$ is the closure of

$$\left\{(x=(x_1,x_2,x_3);\, s) \in \left(\mathbb{C}^3 - \{0\}\right) \times \mathbb{CP}^2 \mid x \sim s,\ x_1 x_2 = x_3^M \right\}.$$

Near the point $(x;s)=(0,0,0;1,0,0)$ we use x_1, s_3 as coordinates and write $(x_1,x_2,\, x_3; s_1,s_2,s_3) = (x_1, x_1^{M-1}s_3^M, x_1 s_3; 1, x_1^{M-2}s_3^M, s_3)$. In this coordinate patch, the Poincaré dual of E_0 is given by the equation $x_1=0$. Let $\kappa \in H^2(T,\mathbb{Z})$ correspond to the cycle ($z_2 - \zeta\varepsilon$ for some M^{th} root of unity ζ, $\varepsilon = const.$), then its image $\sqrt{M}\kappa \in K_M$ corresponds to $(x_1^{M-1}s_3^M - \varepsilon^M)$. So, as $\varepsilon \to 0$, our cycle decomposes into $(M-1)(x_1) + M(s_3)$. In other words, we can calculate the Poincaré dual F of (s_3) from $\sqrt{M}\kappa = (M-1)e + MF$,

where $e \in \Gamma_M$ with $e = E_0 + \cdots$, the dots denoting contributions from further blow ups. This way, we can construct forms F of type iii. for all relevant G.

We conclude that for general G, $H^2(X,\mathbb{Z})$ contains

I. $\pi_*(H^2(T,\mathbb{Z})^G) \cong H^2(T,\mathbb{Z})^G(|G|)$,

II. $\mathcal{E}_{|G|}$, the Poincaré duals of the rational spheres comprising the exceptional divisors in the blow up listed in (2.2),

III. forms of type $\frac{1}{\sqrt{|G|}}\kappa + \frac{1}{|G|}e$, where for the dual of $\kappa \in H^2(T,\mathbb{Z})^G$ we pick a non-generic position on T. Then the fixed point $s \in \mathcal{S}$ occurs on that cycle with multiplicity a_s, and $e = \sum_{s\in\mathcal{S}} a_s E_s$ for an appropriate $E_s \in \Gamma_s$, such that $\frac{1}{|\Gamma_s^*/\Gamma_s|}E_s \in \Gamma_s^*$ is primitive if $\kappa \in H^2(T,\mathbb{Z})^G$ is.

For all but D_4 type singularities $s \in \mathcal{S}$, since Γ_s^*/Γ_s is cyclic, $\frac{1}{|\Gamma_s^*/\Gamma_s|}E_s$ is already determined by III. up to a sign and contributions from II. In all cases, the remaining ambiguities can be cleared up by the fact that $H^2(X,\mathbb{Z})$ is an even lattice. Moreover, in a case by case study we find that as in the Kummer case the vectors listed in I.-III. already generate $H^2(X,\mathbb{Z})$. Namely, those of type III. allow to read off generators of $K_{|G|}^*/K_{|G|}$ and $K_{|G|}/\pi_*(H^2(T,\mathbb{Z})^G)$ (or analogously for $\Pi_{|G|}$); this turns out to determine $K_{|G|}$ already, thus $|K_{|G|}^*/K_{|G|}|$ is known. Now one uses

$$|K_{|G|}^*/K_{|G|}| = |\Pi_{|G|}^*/\Pi_{|G|}| = \frac{\operatorname{disc}\Gamma_{|G|}}{\left[\Pi_{|G|} : \Gamma_{|G|}\right]^2} \tag{2.4}$$

to check that all generators of $\Pi_{|G|}/\Gamma_{|G|}$ have been found.

To illustrate the above recipe we present the case $G = \mathbb{Z}_4$; see Prop. 2.1 for notations. Pick generators $\{\mu_1,\mu_2\}$, $\{\mu_3,\mu_4\}$ of $H^1(T^2,\mathbb{Z})$, $H^1(\widetilde{T}^2,\mathbb{Z})$ with $\mu_i = dx_i$ with respect to coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$. Then $\mu_1 \wedge \mu_2$ is Poincaré dual to $(z_1 - const.)$ and for non-generic $const.$ may contain the fixed points $\{(i,0,0), (i,1,0), (i,0,1), (i,1,1)\}$ with $i \in \mathbb{F}_2^2$. Since for $i = (0,0)$ and $i = (1,1)$ the $\mathbb{Z}_4$ action identifies $(i,1,0)$ with $(i,0,1)$, we find the following lattice vectors of type III.

$$\tfrac{\sqrt{4}}{4}\mu_1 \wedge \mu_2 + \tfrac{1}{4}(2E_{(0,0,1,0)+\varepsilon(1,1,0,0)} + E_{(0,0,0,0)+\varepsilon(1,1,0,0)} + E_{(0,0,1,1)+\varepsilon(1,1,0,0)}),$$

$\varepsilon \in \{0,1\}$. The cycles with $i = (1,0)$ and $i = (0,1)$ must be added to be $\mathbb{Z}_4$ invariant and then give

$$\tfrac{2\sqrt{4}}{4}\mu_1 \wedge \mu_2 + \tfrac{1}{2}(E_{(1,0,0,0)} + E_{(1,0,1,0)} + E_{(1,0,0,1)} + E_{(1,0,1,1)}).$$

The latter vector is spurious as can be seen from the list in Prop. 2.1. The other elements of M_4 in that proposition are obtained analogously from cycles $(z_2 - const.)$, $(\xi_+ + \overline{\xi}_- - const.) + (\xi_+ - \overline{\xi}_- - const.)$ where $\xi_\pm := z_1 \pm i z_2$, and with $\eta_\pm := z_1 \pm z_2$ from $(\eta_+ + \overline{\eta}_- - const.) + (\eta_+ - \overline{\eta}_- - const.)$. The relative signs of the E_i are determined by the fact that $H^2(X,\mathbb{Z})$ is an even lattice.

So far, we have found a set $M_4 \subset H^2(X,\mathbb{Z})$ from which we can read generators of Π_4/Γ_4 as listed in Prop. 2.1. Moreover, we find $\mu_1\wedge\mu_3+\mu_4\wedge\mu_2$, $\mu_1\wedge\mu_4+\mu_2\wedge\mu_3 \in K_4$. We remark that these forms can be used to generate $H^{2,0}(T,\mathbb{Z})$ and $H^{0,2}(T,\mathbb{Z})$ for $T=\mathbb{R}^4/\mathbb{Z}^4$, and by the above observation the transcendental lattice of the corresponding $\mathbb{Z}_4$ orbifold has quadratic form $\mathrm{diag}(2,2)$. This is in agreement with [SI77, Lemma 5.2].

As to the construction of $H^2(X,\mathbb{Z})$, since from M_4 we find $\frac{1}{2}\mu_1\wedge\mu_2$, $\frac{1}{2}\mu_3\wedge\mu_4$, $\frac{1}{2}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2)$, $\frac{1}{2}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3) \in K_4^*$, we conclude

$$K_4 = \mathrm{span}_\mathbb{Z}(2\mu_1\wedge\mu_2, 2\mu_3\wedge\mu_4, \mu_1\wedge\mu_3+\mu_4\wedge\mu_2, \mu_1\wedge\mu_4+\mu_2\wedge\mu_3),$$

and $|K_4^*/K_4| = 4^3$. Hence (2.4) shows $[\Pi_4\colon\Gamma_4]=16$ and proves that the three vectors listed in Prop. 2.1 generate $\Pi_4/\Gamma_4 \cong \mathbb{Z}_4\times\mathbb{Z}_2^2$ and therefore together with $\mathcal{E}_4$ suffice to generate Π_4.

Proposition 2.1
Let X denote an orbifold limit $\widetilde{T/G}$ of $K3$, where G does not contain non-trivial translations. In other words [Fuj88], $G=\mathbb{Z}_M$ with $M\in\{2,3,4,6\}$, $G=\widehat{D}_n$, $n\in\{4,5\}$, $G=\widehat{D}_4'$, or $G=\widehat{\mathbb{T}}$. Then $H^2(X,\mathbb{Z})$ is generated by $\pi_*(H^2(T,\mathbb{Z})^G)\cong H^2(T,\mathbb{Z})^G(|G|)$, the Poincaré duals $\mathcal{E}_{|G|}$ of components of exceptional divisors listed in (2.2), and the set $M_{|G|}$ given by iii. for $G=\mathbb{Z}_2$ or otherwise listed below.

Here, for each type of fixed point $s\in\mathcal{S}$ we fix generators $\frac{1}{|\Gamma_s^*/\Gamma_s|}E_s$ or $\frac{1}{2}E_s^{(a,b)}$ of Γ_s^*/Γ_s with

$$\begin{array}{rrcl}
t\in\mathbb{F}_3^2: & E_t &:=& E_t^{(1)}+2E_t^{(2)};\ i\in I^{(4)}\colon\ E_i:=E_i^{(1)}+2E_i^{(2)}+3E_i^{(3)};\\
G=\mathbb{Z}_6: & E_0^{\mathbb{Z}_6} &:=& E_0^{(1)}+2E_0^{(2)}+3E_0^{(3)}+4E_0^{(4)}+5E_0^{(5)};\\
G=\widehat{D}_4^{(\prime)}: & E_i^{(a,b)} &:=& E_i^{(a)}+E_i^{(b)} \quad (i\in\{0,1\} \text{ or } i\in I^{(4)};\ a,b\in\{1,2,3\});\\
G=\widehat{D}_5: & E_0^{\widehat{D}_5} &:=& 5E_0^{(3)}+6E_0^{(5)}+3E_0^{(4)}+4E_0^{(1)}+2E_0^{(2)};\\
G=\widehat{\mathbb{T}}: & E_0^{E_6} &:=& E_0^{(1)}+2E_0^{(2)}+4E_0^{(4)}+5E_0^{(5)}.
\end{array}$$

With standard basis $\{f_j\}$ of $\mathbb{F}_2^4$ let $P_{jk}:=\mathrm{span}_{\mathbb{F}_2}\{f_j,f_k\}$. Moreover, $\{\mu_j$, $j\in\{1,\dots,4\}\}$ always denotes an appropriate basis of $H^1(T,\mathbb{Z})$ such that

$\mu_j \wedge \mu_k$, $j,k \in \{1,\ldots,4\}$ *generate* $H^2(T,\mathbb{Z})$*. More precisely, if* $\mathbb{Z}_4 \subset G$*, then a generator* $\zeta \in \mathbb{Z}_4$ *acts by*

$$\zeta:\ (\mu_1,\,\mu_2,\,\mu_3,\,\mu_4) \longmapsto (\mu_2,\,-\mu_1,\,-\mu_4,\,\mu_3),$$

and for $G=\mathbb{Z}_3$*,* $\mathbb{Z}_6$ *and* $\widehat{D}_5$ *a* $\mathbb{Z}_3$ *generator* ζ' *acts by*

$$\zeta':\ (\mu_1,\,\mu_2,\,\mu_3,\,\mu_4) \longmapsto (\mu_2-\mu_1,\,-\mu_1,\,-\mu_4,\,\mu_3-\mu_4).$$

We find:

$$M_3 = \left\{ \begin{array}{l} \frac{1}{\sqrt{3}}\mu_1\wedge\mu_2 + \frac{1}{3}\left(E_{(i,0)}+E_{(i,1)}+E_{(i,2)}\right), \quad i\in\mathbb{F}_3, \\ \frac{1}{\sqrt{3}}\mu_3\wedge\mu_4 - \frac{1}{3}\left(E_{(0,i)}+E_{(1,i)}+E_{(2,i)}\right), \quad i\in\mathbb{F}_3, \\ \frac{1}{\sqrt{3}}(\mu_1-\mu_3)\wedge(\mu_2-\frac{1}{2}\mu_1+\mu_4-\frac{1}{2}\mu_3) \\ \qquad\qquad\qquad + \frac{1}{3}\left(E_{(0,0)}+E_{(1,2)}+E_{(2,1)}\right), \\ \frac{1}{\sqrt{3}}(\mu_1-\mu_4)\wedge(\mu_2-\mu_3) + \frac{1}{3}\left(E_{(0,0)}+E_{(1,1)}+E_{(2,2)}\right) \end{array} \right\},$$

generators of Π_3/Γ_3*:*

$$\frac{1}{3}\left(\sum_{t\in L} E_t - \sum_{t'\in L'} E_{t'}\right), \quad L, L' \subset \mathbb{F}_3^2 \text{ parallel lines.}$$

$$M_4 = \left\{ \begin{array}{l} \frac{1}{2}\mu_1\wedge\mu_2 + \frac{1}{2}E_{(0,0,1,0)+\varepsilon(1,1,0,0)} \\ \qquad\qquad + \frac{1}{4}\sum\limits_{i\in P_{34}\cap I^{(4)}} E_{i+\varepsilon(1,1,0,0)}, \varepsilon\in\{0,1\}, \\ \frac{1}{2}\mu_3\wedge\mu_4 - \frac{1}{2}E_{(1,0,0,0)+\varepsilon(0,0,1,1)} \\ \qquad\qquad - \frac{1}{4}\sum\limits_{i\in P_{12}\cap I^{(4)}} E_{i+\varepsilon(0,0,1,1)}, \varepsilon\in\{0,1\}, \\ \frac{1}{2}\left(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2\right) - \frac{1}{2}\sum\limits_{i\in P_{13}} E_{i+j} + E_j, \quad j\in I^{(4)}, \\ \frac{1}{2}\left(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3\right) - \frac{1}{2}\sum\limits_{i\in P_{14}} E_{i+j} + E_j, \quad j\in I^{(4)} \end{array} \right\},$$

generators of Π_4/Γ_4*:*

$$\frac{1}{4}\left(E_{(0,0,0,0)}+E_{(1,1,0,0)}-E_{(0,0,1,1)}-E_{(1,1,1,1)}\right) + \frac{1}{2}\left(E_{(1,0,0,0)}-E_{(1,0,1,1)}\right),$$

$$\frac{1}{2}\left(E_{(0,0,0,0)}+E_{(1,0,0,0)}+E_{(1,0,1,0)}+E_{(1,0,0,1)}-E_{(0,0,1,1)}+E_{(1,0,1,1)}\right),$$

$$\frac{1}{2}\left(E_{(0,0,0,0)}+E_{(0,0,1,0)}+E_{(1,0,1,0)}+E_{(1,0,0,1)}-E_{(1,1,0,0)}+E_{(1,1,1,0)}\right).$$

$$M_6 = \left\{ \begin{array}{l} \frac{1}{\sqrt{6}}\mu_1\wedge\mu_2 + \frac{1}{6}E_0^{\mathbb{Z}_6} + \frac{1}{3}E_{(0,1)} + \frac{1}{2}E_{(0,0,0,1)}, \\ \frac{1}{\sqrt{6}}\mu_3\wedge\mu_4 - \frac{1}{6}E_0^{\mathbb{Z}_6} - \frac{1}{3}E_{(1,0)} - \frac{1}{2}E_{(1,0,0,0)}, \\ \frac{2}{\sqrt{6}}\mu_1\wedge\mu_2 + \frac{1}{3}\left(E_{(1,0)} + E_{(1,1)} + E_{(1,2)}\right), \\ \frac{2}{\sqrt{6}}\mu_3\wedge\mu_4 - \frac{1}{3}\left(E_{(0,1)} + E_{(1,1)} + E_{(1,2)}\right), \\ \frac{3}{\sqrt{6}}\mu_1\wedge\mu_2 + \frac{1}{2}\left(E_{(1,0,0,0)} + E_{(1,0,1,0)} + E_{(1,0,0,1)} + E_{(0,1,0,1)}\right), \\ \frac{3}{\sqrt{6}}\mu_3\wedge\mu_4 - \frac{1}{2}\left(E_{(0,0,0,1)} + E_{(1,0,1,0)} + E_{(1,0,0,1)} + E_{(0,1,0,1)}\right), \\ \frac{1}{\sqrt{6}}(\mu_1-\mu_3)\wedge(\mu_2 - \frac{1}{2}\mu_1 + \mu_4 - \frac{1}{2}\mu_3) \\ \qquad\qquad\qquad + \frac{1}{6}E_0^{\mathbb{Z}_6} + \frac{1}{3}E_{(1,2)} + \frac{1}{2}E_{(1,0,0,1)}, \\ \frac{1}{\sqrt{6}}(\mu_1-\mu_4)\wedge(\mu_2-\mu_3) + \frac{1}{6}E_0^{\mathbb{Z}_6} + \frac{1}{3}E_{(1,1)} + \frac{1}{2}E_{(0,1,0,1)} \end{array} \right\},$$

generator of Π_6/Γ_6*:*

$$\frac{1}{6}E_0^{\mathbb{Z}_6} + \frac{1}{3}\left(E_{(1,0)} + E_{(0,1)} + E_{(1,1)} + E_{(1,2)}\right) + \frac{1}{2}\left(E_{(1,0,0,0)} + E_{(0,0,0,1)} + E_{(1,0,1,0)} + E_{(1,0,0,1)} + E_{(0,1,0,1)}\right).$$

$$M_8 = \left\{ \begin{array}{l} \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4) - \frac{1}{2}E_{(1,0,0,0)} - \frac{1}{4}E_{(1,1,0,0)} - \frac{1}{2}E_0^{(1,2)}, \\ \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4) + \frac{1}{2}E_{(0,1,1,1)} + \frac{1}{4}E_{(1,1,0,0)} + \frac{1}{2}E_1^{(1,2)}, \\ \frac{2}{\sqrt{8}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4) - \frac{1}{2}\sum_{i\in\mathbb{F}_2^2} E_{(1,0,i)}, \\ \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2) - \frac{1}{2}E_{(1,0,0,0)} - \frac{1}{4}E_{(1,0,1,0)} - \frac{1}{2}E_0^{(1,3)}, \\ \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2) + \frac{1}{2}E_{(0,1,1,1)} + \frac{1}{4}E_{(1,0,1,0)} + \frac{1}{2}E_1^{(1,3)}, \\ \frac{2}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2) - \frac{1}{2}\sum_{i_1,i_2\in\mathbb{F}_2} E_{(i_1,1,i_2,0)}, \\ \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3) - \frac{1}{2}E_{(1,0,0,0)} - \frac{1}{4}E_{(1,0,0,1)} - \frac{1}{2}E_0^{(2,3)}, \\ \frac{1}{\sqrt{8}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3) + \frac{1}{2}E_{(0,1,1,1)} + \frac{1}{4}E_{(1,0,0,1)} + \frac{1}{2}E_1^{(2,3)}, \\ \frac{2}{\sqrt{8}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3) - \frac{1}{2}\sum_{i\in\mathbb{F}_2^2} E_{(1,i,0)} \end{array} \right\},$$

generators of Π_8/Γ_8*:*

$$\begin{aligned}
&\tfrac{1}{2}\left(E_{(1,0,0,0)}+E_{(0,1,1,1)}+E_{(1,1,0,0)}+E_0^{(1,2)}+E_1^{(1,2)}\right),\\
&\tfrac{1}{2}\left(E_{(1,0,0,0)}+E_{(0,1,1,1)}+E_{(1,0,1,0)}+E_0^{(1,3)}+E_1^{(1,3)}\right),\\
&\tfrac{1}{2}\left(E_{(1,0,0,0)}+E_{(0,1,1,1)}+E_{(1,0,0,1)}+E_0^{(2,3)}+E_1^{(2,3)}\right).
\end{aligned}$$

$$M_8'=\left\{\begin{array}{l}
\frac{1}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+2\mu_4\wedge\mu_3)\\
\qquad\qquad+\tfrac{1}{2}\left(E^{(1,2)}_{(0,0,0,0)}+E^{(1,2)}_{(1,1,0,0)}+E_{(0,0,0,1)}\right),\\
\frac{1}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+2\mu_4\wedge\mu_3)\\
\quad+\tfrac{1}{2}\left(E^{(1,2)}_{(0,0,0,0)}+E^{(2,3)}_{(1,1,0,0)}+E^{(2,3)}_{(0,0,1,1)}+E^{(1,2)}_{(1,1,1,1)}+E_{(0,0,0,1)}\right),\\
\frac{1}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+2\mu_4\wedge\mu_3)\\
\quad+\tfrac{1}{2}\left(E^{(1,2)}_{(0,0,0,0)}+E^{(1,3)}_{(1,1,0,0)}+E^{(1,2)}_{(0,0,1,1)}+E^{(1,3)}_{(1,1,1,1)}+E_{(0,0,0,1)}\right),\\
\frac{2}{\sqrt{8}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4)+\tfrac{1}{2}\left(E_{(1,0,0,0)}+E_{(0,0,0,1)}\right),\\
\frac{2}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_4\wedge\mu_3)+\tfrac{1}{2}\left(E_{(1,0,1,0)}+E_{(0,0,0,1)}\right),\\
\frac{2}{\sqrt{8}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_4\wedge\mu_3)+\tfrac{1}{2}\left(E_{(1,0,1,0)}+E_{(0,0,0,1)}\right)\\
\qquad\qquad+\tfrac{1}{2}\sum_{i\in I^{(4)}}E_i^{(a,b)},\qquad a,b\in\{1,2,3\},a\neq b,\\
\frac{2}{\sqrt{8}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+\mu_4\wedge\mu_3)+\tfrac{1}{2}\left(E_{(1,0,1,0)}+E_{(0,0,0,1)}\right),\\
\frac{2}{\sqrt{8}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+\mu_4\wedge\mu_3)+\tfrac{1}{2}\left(E_{(1,0,1,0)}+E_{(0,0,0,1)}\right)\\
\qquad\qquad+\tfrac{1}{2}\sum_{i\in I^{(4)}}E_i^{(a,b)},\qquad a,b\in\{1,2,3\},a\neq b
\end{array}\right\},$$

generators of Π_8'/Γ_8'*:*

$$\tfrac{1}{2}\left(E^{(1,3)}_{(1,1,0,0)}+E^{(2,3)}_{(0,0,1,1)}+E^{(1,2)}_{(1,1,1,1)}\right),\ \ \tfrac{1}{2}\left(E^{(2,3)}_{(1,1,0,0)}+E^{(1,2)}_{(0,0,1,1)}+E^{(1,3)}_{(1,1,1,1)}\right),$$
$$\tfrac{1}{2}\sum_{i\in I^{(4)}}E_i^{(a,b)},\quad a,b\in\{1,2,3\},\ a\neq b.$$

$$M_{12}=\left\{\begin{array}{l}\frac{1}{\sqrt{12}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4)-\frac{1}{2}E_0^{\widehat{D}_5}-\frac{1}{3}E_{(1,0)}-\frac{1}{2}E_{(1,0,0,0)},\\ \frac{3}{\sqrt{12}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4)-\frac{1}{2}\sum_{i\in\mathbb{F}_2^2}E_{(1,0,i)},\\ \frac{2}{\sqrt{12}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4)+\frac{1}{3}E_{(1,0)},\\ \frac{2}{\sqrt{12}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+\mu_3\wedge\mu_1)\\ \qquad\qquad\qquad\qquad -\frac{1}{2}E_0^{\widehat{D}_5}-\frac{1}{3}E_{(1,1)}-\frac{1}{2}E_{(1,0,1,0)},\\ \frac{2}{\sqrt{12}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2)-\frac{1}{2}E_0^{\widehat{D}_5}-\frac{1}{3}E_{(1,1)}-\frac{1}{2}E_{(1,0,0,1)}\end{array}\right\},$$

generator of Π_{12}/Γ_{12}:

$$\tfrac{1}{2}\left(E_0^{\widehat{D}_5}+E_{(1,0,1,0)}+E_{(1,0,0,1)}+E_{(0,1,0,1)}\right).$$

$$M_{24}=\left\{\begin{array}{l}\frac{3}{\sqrt{24}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+2\mu_4\wedge\mu_3)\\ \qquad\qquad\qquad +\frac{1}{2}\Big(E^{(1,2)}_{(1,1,0,0)}+E_{(0,0,0,1)}\Big),\\ \frac{6}{\sqrt{24}}(\mu_1\wedge\mu_2+\mu_3\wedge\mu_4)+\frac{1}{2}E^{(1,3)}_{(1,1,0,0)},\\ \frac{6}{\sqrt{24}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_4\wedge\mu_3)\\ \qquad\qquad\qquad +\frac{1}{3}E_0^{E_6}+\frac{1}{3}E_{(1,1)}+\frac{1}{2}E^{(2,3)}_{(1,1,0,0)},\\ \frac{6}{\sqrt{24}}(\mu_1\wedge\mu_3+\mu_4\wedge\mu_2+\mu_4\wedge\mu_3)\\ \qquad\qquad\qquad -\frac{1}{3}\left(E_{(1,0)}+E_{(0,1)}+E_{(1,2)}\right)+\frac{1}{2}E^{(2,3)}_{(1,1,0,0)},\\ \frac{6}{\sqrt{24}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+\mu_4\wedge\mu_3)\\ \qquad\qquad\qquad +\frac{1}{3}E_0^{E_6}+\frac{1}{3}E_{(1,1)}+\frac{1}{2}E^{(2,3)}_{(1,1,0,0)},\\ \frac{6}{\sqrt{24}}(\mu_1\wedge\mu_4+\mu_2\wedge\mu_3+\mu_4\wedge\mu_3)\\ \qquad\qquad\qquad -\frac{1}{3}\left(E_{(1,0)}+E_{(0,1)}+E_{(1,2)}\right)+\frac{1}{2}E^{(2,3)}_{(1,1,0,0)},\end{array}\right\},$$

generator of Π_{24}/Γ_{24}:

$$\tfrac{1}{3}E_0^{E_6}+\tfrac{1}{3}\left(E_{(1,1)}+E_{(1,0)}+E_{(0,1)}+E_{(1,2)}\right).$$

Prop. 2.1 pinpoints the characteristic distinction between our discussion of compact orbifolds as opposed to the approach of [Dou97, BI97]. On $Y = \widetilde{\mathbb{C}^2/G}$, the components of exceptional divisors generate $H_2(Y,\mathbb{Z})$, whereas on our $K3$ surface X it is a nontrivial problem to determine the primitive sublattices of $H_2(X,\mathbb{Z})$ that contain the exceptional divisors.

3 Consistent embedding of orbifold conformal field theories in $\mathcal{M}^{K3}$

Prop. 2.1 serves to explicitly locate G orbifold limits of Einstein metrics of volume 1 on $K3$ within the moduli space

$$O^+(H^2(X,\mathbb{Z}))\backslash O^+(H^2(X,\mathbb{R}))/SO(3)\times O(19)$$

of such metrics. Analogous reasoning should enable us to locate G orbifold conformal field theories in the moduli space

$$O^+(H^{even}(X,\mathbb{Z}))\backslash O^+(H^{even}(X,\mathbb{R}))/SO(4)\times O(20)$$

of conformal field theories associated to $K3$. Again, the construction of the appropriate embedding $\widehat{\pi}_*(H^{even}(T,\mathbb{Z})^G)\hookrightarrow H^{even}(X,\mathbb{Z})$ is all we need, where $\widehat{\pi}_*$ is an extension of π_*. Hence we must require the image $x\subset H^{even}(X,\mathbb{R})$ of a four–plane $x_T\in\mathcal{T}^{4,4}$, which describes a toroidal conformal field theory with G symmetry, to admit a geometric interpretation on the corresponding G orbifold limit of $K3$. This statement is made precise by the use of (B.1), which assigns a geometric interpretation to any of our conformal field theories:

Suppose $v,v^0\in H^{even}(T,\mathbb{Z})$ are primitive null vectors with $\langle v,v^0\rangle=1$ such that x_T has geometric interpretation (Σ_T,V_T,B_T). We need to find primitive null vectors $\widehat{v},\widehat{v}^0\in H^{even}(X,\mathbb{Z})$ with $\langle\widehat{v},\widehat{v}^0\rangle=1$ such that x has geometric interpretation (Σ,V,B), where $\Sigma=\pi_*\Sigma_T$. The location of this three–plane in $H^2(X,\mathbb{R})$ is fixed by Prop. 2.1. In particular, for each G orbifold, the two–forms corresponding to exceptional divisors of the blow up must be contained in $H^{even}(X,\mathbb{Z})$ in such a way that the exceptional divisors have volume zero:

$$\operatorname{span}_{\mathbb{R}}(K_{|G|},\widehat{v},\widehat{v}^0)^\perp\cap H^{even}(X,\mathbb{Z})\supset\widehat{\Pi}_{|G|}\cong\Pi_{|G|}. \tag{3.1}$$

Let $\Lambda_{|G|}$ denote the primitive sublattice of $H^{even}(X,\mathbb{Z})$ which contains the lattice $\widehat{\pi}_*(H^{even}(T,\mathbb{Z})^G)$. Then $\Lambda_{|G|}\cong K_{|G|}\oplus U(|G|)$, where $U(|G|)$ is generated by the $\widehat{\pi}_*$ images $\sqrt{|G|}v,\sqrt{|G|}v^0$ of v,v^0. Any ansatz with non–primitive $\sqrt{|G|}v$ or $\sqrt{|G|}v^0$ leads to contradictions by the methods presented

below, cf. our comment below Theorem 3.3. Here, $\sqrt{|G|}\upsilon$ is the Poincaré dual of a generic point on X, and $\sqrt{|G|}\upsilon^0$ denotes the dual of the cycle obtained as closure of $\pi(T-\widetilde{\mathcal{S}})$ on X; this interpretation is in accord with $\langle\sqrt{|G|}\upsilon, \sqrt{|G|}\upsilon^0\rangle = |G|$, since π is $|G|$:1 outside the set $\mathcal{S}$ of fixed points. The ad hoc assignment of equal scaling factors $\sqrt{|G|}$ to both υ and υ^0 is chosen for ease of notation; this freedom of choice drops out in all results below, see (3.7).

It follows that we cannot use $\sqrt{|G|}\upsilon, \sqrt{|G|}\upsilon^0$ for $\widehat{\upsilon}, \widehat{\upsilon}^0$:

$$(\Lambda_{|G|})^*/\Lambda_{|G|} \;\cong\; (K_{|G|})^*/K_{|G|} \times \mathbb{Z}_{|G|}^2 \;\cong\; (\Pi_{|G|})^*/\Pi_{|G|} \times \mathbb{Z}_{|G|}^2 \qquad (3.2)$$

by Theorem A.2, so again by Theorem A.2, $(\Lambda_{|G|})^\perp \cap H^{even}(X,\mathbb{Z}) \ncong \Pi_{|G|}$.

Since $\sqrt{|G|}\upsilon$ has a good geometric interpretation as Poincaré dual of the generic point on X, we use the ansatz

$$\widehat{\upsilon} := \sqrt{|G|}\,\upsilon, \quad \widehat{\upsilon}^0 := \tfrac{1}{\sqrt{|G|}}\upsilon^0 - \tfrac{1}{|G|}B_{|G|} - \tfrac{\|B_{|G|}\|^2}{2|G|^2}\sqrt{|G|}\upsilon, \qquad (3.3)$$

with $B_{|G|} \perp \upsilon, \upsilon^0$ to be determined. By [LP81, Nik80b] the automorphism group $O^+(H^{even}(X,\mathbb{Z}))$ in (1.1) acts transitively on pairs of primitive lattice vectors of equal length. Hence (3.3) is also the most general ansatz we need. Assume that for given G we have found $B_{|G|}$ such that (3.1) holds for $\widehat{\upsilon}, \widehat{\upsilon}^0$ as in (3.3) (we will show that $B_{|G|}$ is uniquely determined up to lattice automorphisms). All calculations below are carried out in $H^{even}(X,\mathbb{Q})$. By $Y : H^{even}(X,\mathbb{Q}) \longrightarrow H^{even}(X,\mathbb{Q})$ we denote the orthogonal projection onto $\upsilon^\perp \cap (\upsilon^0)^\perp$ and (by a slight abuse of notation; see the footnote on page 5) set $\Pi_{|G|} := Y(\widehat{\Pi}_{|G|})$, because indeed $\Pi_{|G|} \cong \widehat{\Pi}_{|G|}$ since $\widehat{\Pi}_{|G|} \perp \upsilon$. Then

Lemma 3.1
Suppose that $\widehat{\upsilon}, \widehat{\upsilon}^0 \in H^{even}(X,\mathbb{Z})$ have the form (3.3) and that for every G orbifold conformal field theory determined by $x = \widehat{\pi}_* x_T$ they give a geometric interpretation on the corresponding G orbifold limit of $K3$ with (3.1). Then

$$\begin{aligned} B_{|G|} \in \Pi_{|G|}, \quad \|\tfrac{1}{|G|}B_{|G|}\|^2 &\in 2\mathbb{Z}, \\ \langle B_{|G|}, E\rangle &\equiv -1 \mod |G| \text{ for some } E \in \Pi_{|G|}. \end{aligned}$$

Set $\widehat{M}_{|G|} := M_{|G|} \cup \{\widehat{\upsilon}, \widehat{\upsilon}^0\}$ with $M_{|G|}$ as defined in Prop. 2.1, and

$$\forall\, E \in \Pi_{|G|}: \quad \widehat{E} := E - \langle E, \widehat{\upsilon}^0\rangle\,\widehat{\upsilon} = E + \tfrac{1}{\sqrt{|G|}}\langle B_{|G|}, E\rangle\,\upsilon.$$

Then $\widehat{M}_{|G|}$ and $\widehat{\mathcal{E}}_{|G|} := \{\widehat{E} \mid E \in \Pi_{|G|}\}$ generate $H^{even}(X,\mathbb{Z}) \cong \Gamma^{4,20}$.

Proof:
By Theorem A.2, we need to find $P_{|G|} := (\Lambda_{|G|})^\perp \cap H^{even}(X,\mathbb{Z})$, then $\Lambda^*_{|G|}/\Lambda_{|G|} \cong P^*_{|G|}/P_{|G|}$, with isomorphism denoted by γ, and the discriminant forms agree up to a sign. This will give

$$H^{even}(X,\mathbb{Z}) \cong \left\{ (x,y) \in \Lambda^*_{|G|} \oplus P^*_{|G|} \,|\gamma(\overline{x}) = \overline{y} \right\}. \tag{3.4}$$

We claim that $P_{|G|} = \widehat{P}_{|G|}$ with

$$\widehat{P}_{|G|} := \left\{ p \in \Pi_{|G|} \mid \langle p, \widehat{v}^0 \rangle \in \mathbb{Z} \right\}.$$

Namely, for $p \in \widehat{P}_{|G|}$ by construction we can find $\widetilde{p} \in H^{even}(X,\mathbb{Z})$ such that $\widetilde{p} - p = av$, $a \in \mathbb{R}$. Since $\mathbb{Z} \ni \langle \widetilde{p}, \widehat{v}^0 \rangle = \frac{a}{\sqrt{|G|}} + \langle p, \widehat{v}^0 \rangle$, a is an integral multiple of $\sqrt{|G|}$, and therefore $\widehat{P}_{|G|} \subset H^{even}(X,\mathbb{Z})$. But $P_{|G|} \otimes \mathbb{R} = \widehat{P}_{|G|} \otimes \mathbb{R}$ is clear from $P_{|G|} \subset \Pi_{|G|}$ on dimensional grounds, so $P_{|G|} = \widehat{P}_{|G|}$ since both are primitive sublattices of $H^{even}(X,\mathbb{Z})$ by construction.

From Theorem A.2 we conclude that $B_{|G|}$ must be chosen such that $\widehat{P}^*_{|G|}/\widehat{P}_{|G|} \cong \Lambda^*_{|G|}/\Lambda_{|G|}$ with discriminant forms of opposite sign. Because $\widehat{P}_{|G|} \subset \Pi_{|G|} \subset \Pi^*_{|G|} \subset \widehat{P}^*_{|G|}$, we can use the decomposition

$$\widehat{P}^*_{|G|}/\widehat{P}_{|G|} \cong \widehat{P}^*_{|G|}/\Pi^*_{|G|} \times \Pi^*_{|G|}/\Pi_{|G|} \times \Pi_{|G|}/\widehat{P}_{|G|}, \tag{3.5}$$

so from (3.2) we deduce

$$\Pi_{|G|}/\widehat{P}_{|G|} \cong \widehat{P}^*_{|G|}/\Pi^*_{|G|} \cong \mathbb{Z}_{|G|}.$$

Moreover, $\frac{1}{|G|}B_{|G|}$ generates $\widehat{P}^*_{|G|}/\Pi^*_{|G|}$, thus $B_{|G|} \in \Pi^*_{|G|}$. Since the quadratic forms of $\widehat{P}^*_{|G|}/\widehat{P}_{|G|}$ and $\Lambda^*_{|G|}/\Lambda_{|G|}$ agree up to a sign as forms with values in $\mathbb{Q}/2\mathbb{Z}$, we conclude $\|\frac{1}{|G|}B_{|G|}\|^2 \in 2\mathbb{Z}$, and by (3.5) there exists $E \in \Pi_{|G|}$ which generates $\Pi_{|G|}/\widehat{P}_{|G|}$ such that $\langle B_{|G|}, E \rangle \equiv -1 \mod |G|$. Furthermore, by (3.3) $B_{|G|} \in P_{|G|} = \widehat{P}_{|G|} \subset \Pi_{|G|}$. The generators of $H^{even}(X,\mathbb{Z})$ can now be read off from (3.4) and Prop. 2.1. $\square$

The properties listed in Lemma 3.1 do not determine $B_{|G|}$ in (3.3) uniquely. But since a shift of $\widehat{v}^0$ by an element of $H^{even}(X,\mathbb{Z})$ corresponds to an integral shift of the B–field in the geometric interpretation and thus is irrelevant to our discussion (see (1.1)), we can restrict ourselves to a finite number of candidates for $B_{|G|}$. A lot of them will be equivalent by lattice automorphisms in $O^+(H^2(X,\mathbb{Z}))$. The lift $\widehat{B}_{|G|}$ of $B_{|G|} \in \Pi_{|G|}$ to $\widehat{\Pi}_{|G|}$ will determine the offset $\frac{1}{|G|}\widehat{B}_{|G|}$ of the B–field induced on the exceptional divisors

of the blow up by the orbifold process (see (3.7)). Since this is a local effect for each fixed point, the result for $G=\mathbb{Z}_2$ will determine the contribution of $\mathbb{Z}_2$ fixed points for all G etc.

Moreover, algebraic symmetries of the underlying toroidal conformal field theory induce symmetries of the orbifold conformal field theory that must not be destroyed by the B–field. In particular, $\widehat{B}_{|G|}$ is invariant under all algebraic automorphisms of the orbifold limit of $K3$. For $G=\mathbb{Z}_2$ and $G=\mathbb{Z}_4$ we can use the results [NW01, Thms. 2.7, 2.12] on algebraic automorphisms of orbifold conformal field theories obtained from toroidal theories on $T=\mathbb{R}^4/\mathbb{Z}^4$ to verify that all $\mathbb{Z}_2$ and all $\mathbb{Z}_4$ type fixed points are related by symmetries, respectively, and therefore confirm that they must give the same contribution to $B_{|G|}$. Moreover, in the $\mathbb{Z}_4$ case, all $E_i^{(1;3)}$, $i\in I^{(4)}$, must carry the same B–field flux. Analogous reasoning severely restricts the number of candidates for $B_{|G|}$ in all cases. Actually,

Lemma 3.2
For $G\subset SU(2)$ as in (2.2), the vector $B_{|G|}$ in (3.3) is uniquely fixed, up to lattice automorphisms in $O^+(H^2(X,\mathbb{Z}))$ and shifts of $\widehat{\upsilon}^0$ by lattice vectors, by the properties listed in Lemma 3.1 and consistency with symmetries of G orbifold conformal field theories.

For a $G'\subset G$ type fixed point $s\in\mathcal{S}$ let $\sum_j n_s^{(j)}E_s^{(j)}$ denote the highest root in Γ_s. Then we can characterize $B_{|G|}$ by

$$\forall\, s\in\mathcal{S},\ \forall\, j:\quad \frac{1}{|G|}\langle B_{|G|},E_s^{(j)}\rangle=\frac{n_s^{(j)}}{|G'|}.$$

Proof:
The result $B_2=-\frac{1}{2}\sum_{i\in\mathbb{F}_2^4}E_i$ for $G=\mathbb{Z}_2$ follows immediately from Lemma 3.1 together with the observation that all fixed points contribute equally.

We only add the proof for $G=\mathbb{Z}_4$, since the other cases are obtained analogously. The most general ansatz for $B_4\in\Pi_4$ that is consistent with the symmetries of the $\mathbb{Z}_4$ orbifold of the toroidal model on $\mathbb{R}^4/\mathbb{Z}^4$ and our knowledge of the B–field over the $\mathbb{Z}_2$ fixed points is

$$B_4=-\sum_{i\in I^{(2)}}E_i-\tfrac{\alpha}{2}\sum_{i\in I^{(4)}}(E_i^{(1)}+E_i^{(3)})-\beta\sum_{i\in I^{(4)}}E_i^{(2)},$$

where we can restrict to $\alpha,\beta\in\{0,\dots,3\}$. Then $\frac{\|B_4\|^2}{32}\in\mathbb{Z}$, which must hold by Lemma 3.1, iff $(\alpha,\beta)\in\{(1,2),(1,3),(3,1),(3,2)\}$. For $(\alpha,\beta)=(3,1)$ there is no $E\in\Pi_4$ with $\langle B_4,E\rangle\equiv-1 \mod 4$. We claim that the remaining three cases are equivalent by lattice automorphisms in $O^+(H^2(X,\mathbb{Z}))$.

Indeed, $(\alpha,\beta)=(1,3)$ turns into $(\alpha,\beta)=(1,2)$ by

$$E_i^{(1;3)} \longmapsto -E_i^{(1;3)} - E_i^{(2)}, \qquad E_i^{(2)} \longmapsto E_i^{(1)} + E_i^{(2)} + E_i^{(3)},$$

and $(\alpha,\beta)=(1,2)$ turns into $(\alpha,\beta)=(3,2)$ by

$$E_i^{(1;3)} \longmapsto -E_i^{(1;3)}, \qquad E_i^{(2)} \longmapsto E_i^{(1)} + E_i^{(2)} + E_i^{(3)}.$$

Both maps induce identity on Π_4^*/Π_4 and hence can be trivially continued to elements of $O^+(H^2(X,\mathbb{Z}))$ by [Nik80a, Prop. 1.1]. $\square$

Lemmata 3.1, 3.2, and Theorem B.1 determine the desired embedding $\widehat{\pi}_*(H^{even}(T,\mathbb{Z})^G) \hookrightarrow H^{even}(X,\mathbb{Z})$. Note that apart from the observation that $\frac{1}{2}\sum_{i\in\mathbb{F}_2^4} E_i \in \Pi_2$, $\frac{1}{2}\sum_{i\in I^{(4)}}(E_i^{(1)}+E_i^{(3)}) \in \Pi_4$ etc. the result for $B_{|G|}$ is independent of the explicit calculations that led to Prop. 2.1. It is crucial to understand that for $E \in \Pi$ we found that in general $E \notin H^{even}(X,\mathbb{Z})$. Lemma 3.1 shows that there is a lift $\widehat{E} \in H^{even}(X,\mathbb{Z})$ for every such vector. In particular, only $\widehat{E}$ can have a geometrical meaning. We lift

$$\begin{array}{rcl} \widehat{B}_{|G|} := B_{|G|} - \langle B_{|G|}, \widehat{\upsilon}^0\rangle\widehat{\upsilon} & = & B_{|G|} + \frac{\|B_{|G|}\|^2}{|G|}\widehat{\upsilon} \\ & = & B_{|G|} - 2|G|\,\widehat{\upsilon} \quad \in \quad H^{even}(X,\mathbb{Z}) \end{array} \tag{3.6}$$

to find that for generators of $\widehat{\pi}_* x_T$

$$\begin{array}{rcl} \text{with} \quad B := \frac{1}{\sqrt{|G|}}B_T + \frac{1}{|G|}\widehat{B}_{|G|} : & & \\ \forall\, \sigma \in \Sigma_T : \qquad \sigma - \langle \sigma, B_T\rangle \upsilon & = & \sigma - \langle \sigma, B\rangle \widehat{\upsilon}, \\ \frac{1}{\sqrt{|G|}}\left(\upsilon^0 + B_T + \left(V_T - \frac{\|B_T\|^2}{2}\right)\upsilon\right) & = & \widehat{\upsilon}^0 + B + \left(\frac{V_T}{|G|} - \frac{\|B\|^2}{2}\right)\widehat{\upsilon}. \end{array} \tag{3.7}$$

Compare with Theorem B.1 to see that this proves

Theorem 3.3
Let (Σ_T, V_T, B_T) denote a geometric interpretation of a toroidal nonlinear sigma model on the torus T that admits a G symmetry, $G \subset SU(2)$ not containing non-trivial translations; all such G actions are specified in (2.2). Then its image $x \in \mathcal{T}^{4,20}$ under the G orbifold procedure has geometric interpretation (Σ, V, B) where $\Sigma \in \mathcal{T}^{3,19}$ is found as described by Prop. 2.1, $V = \frac{V_T}{|G|}$, and $B = \frac{1}{\sqrt{|G|}}B_T + \frac{1}{|G|}\widehat{B}_{|G|}$, $\widehat{B}_{|G|} \in H^{even}(X,\mathbb{Z})$ as in (3.6) with $\langle B_{|G|}, E_s^{(j)}\rangle = \langle \widehat{B}_{|G|}, \widehat{E}_s^{(j)}\rangle$ the $|G{:}G'|$–fold coefficient of $E_s^{(j)}$ in the highest root of Γ_s, $s \in \mathcal{S}$ a $G' \subset G$ type fixed point (see Lemma 3.2).

With (3.7) we can confirm that the primitiveness of $\sqrt{|G|}\upsilon, \sqrt{|G|}\upsilon^0$ in

$H^{even}(X,\mathbb{Z})$ indeed follows from our requirement that orbifold conformal field theories are consistently embedded in $\mathcal{M}^{K3}$. This can either be seen from consistency with the symmetry $\upsilon \leftrightarrow \upsilon^0$ on $\mathcal{M}^{tori}$ or the rescaling of B_T under the embedding that follows from (3.7). Namely, assuming $\frac{1}{\sqrt{|G|}}\upsilon^0 \in H^{even}(X,\mathbb{Z})$ and hence $B_{|G|} = \widehat{B}_{|G|} = 0$ in (3.3), (3.7) directly leads to a contradiction: B_T and $B_T + \lambda$, $\lambda \in H^2(T,\mathbb{Z})$, give equivalent torus theories by (1.1), but $B = B_T/\sqrt{|G|}$, and $B = (B_T + \lambda)/\sqrt{|G|}$ in general do not give equivalent $K3$ theories. Not so if we use (3.7) with $\widehat{B}_{|G|}$ as determined above, by the results of Prop. 2.1, since then the shift by λ can be compensated by a lattice automorphism. Any other ansatz with non-primitive $\sqrt{|G|}\upsilon$, $\sqrt{|G|}\upsilon^0$ leads to an analogous contradiction by the methods presented in Lemma 3.2.

4 Discussion

Let us summarize the results of this work: By Prop. 2.1 and Theorem 3.3, for all orbifold constructions of $K3$ obtained from non-translationary groups, the precise location of the corresponding orbifold conformal field theories within the moduli space $\mathcal{M}^{K3}$ of theories associated to $K3$ has been determined. To arrive at Prop. 2.1, we have presented a technique to calculate the generalization of the Kummer lattice to all these orbifolds. The fact that the components of the exceptional divisors of the blow up do not generate primitive sublattices of $H_2(X,\mathbb{Z})$ distinguishes our compact X from the minimal resolution of $\mathbb{C}^2/G$. The explicit results listed in Prop. 2.1 should allow a detailed analysis of D–branes on orbifold limits of $K3$ in the spirit of [RW98, BER99]. Sect. 3 contains an elementary new proof for the fact that the orbifold procedure forces fixed values on the B–field of the orbifold conformal field theory in direction of the exceptional divisors [Asp95]. Our proof is mostly independent of the technical discussion in Sect. 2. It merely uses the known description of the moduli space $\mathcal{M}^{K3}$ [AM94] and shows that the B–field flux can be interpreted as artifact from a consistent embedding of orbifold conformal field theories in $\mathcal{M}^{K3}$. We also prove that our consistency requirement already fixes the B–field values uniquely up to lattice automorphisms, and we are able to read them off explicitly. For the cyclic groups, we are in agreement with [Asp95] ($G = \mathbb{Z}_2$), and with [Dou97, BI97] where mass formulae and tadpole cancellation conditions for D–branes were used; the author did not find the explicit results for the binary dihedral and tetrahedral groups in the literature.

Theorem 3.3 indicates a connection to the classical McKay correspondence [McK80, McK81], which has also inspired very recent work in the

physics literature [DD, GJ01, Tom01, May01]. All of the latter publications concentrate on higher dimensional cases where large volume limits are used, though. Consider the local picture near any of our fixed points $s \in \mathcal{S}$. Without loss of generality $s = 0$, and we are studying the minimal resolution Y of $\mathbb{C}^2/G', G' \subset G$. By Theorem 3.3, the contribution from this fixed point to $\widehat{B}_{|G|}$ is $|G\!:\!G'|\,\widehat{B}^s_{|G|}$ with $\widehat{B}^s_{|G|} := \sum_j n_s^{(j)} (\widehat{E}_s^{(j)})^*$, where $\sum_j n_s^{(j)} \widehat{E}_s^{(j)}$ is the highest root of $\widehat{\Gamma}_s$, and $\{(\widehat{E}_s^{(j)})^*\} \subset \widehat{\Gamma}_s \otimes \mathbb{Q}$ denotes the dual basis of the fundamental system $\{\widehat{E}_s^{(j)}\}$ of $\widehat{\Gamma}_s$. Recall that the j^{th} node in the extended Dynkin diagram of G' labels an irreducible representation ρ_j of G' of dimension $n_s^{(j)}$, where $n_s^{(0)} = 1$, and ρ_0 is the trivial representation. On the other hand, the McKay correspondence as proven in [GSV83, Knö85, AV85] states that $(\widehat{E}_s^{(j)})^*$ is the first Chern class of a locally free sheaf on Y that is built from the associated bundle on $\mathbb{C}^2/G' - \{0\}$ given by ρ_j. Since the regular representation ρ of G' decomposes as $\rho = \sum_j n_s^{(j)} \rho_j$, $\widehat{B}^s_{|G|}$ is the first Chern class of the extension to Y of $\pi_* \mathcal{O}_{\mathbb{C}^2 - \{0\}}(\mathbb{C}^{|G'|})$ with regular G' action on $\mathbb{C}^{|G'|}$. It appears reasonable to assume that similarly to [Kob90] the construction of [GSV83, Knö85] can be carried over to X by gluing appropriate sheaves near each fixed point in a deformation of X and taking the orbifold limit. For the present case this in fact follows from the results in [BKR99]. Then $\widehat{B}_{|G|}$ is the first Chern class of a sheaf $\mathcal{E} \to X$ obtained from $\pi_* \mathcal{O}_{T - \widehat{\mathcal{S}}}(\mathbb{C}^{|G|})$ by continuation. For $G = \mathbb{Z}_M, M \in \{2,3,4,6\}$, we find that the corresponding Mukai vector [Muk84, Muk87] obeys

$$
\begin{aligned}
ch(\mathcal{E})\sqrt{\widehat{A}(X)} &= [\mathrm{rk}\ \mathcal{E}]\,\widehat{\upsilon}^0 + c_1(\mathcal{E}) + \left[(c_2 - \tfrac{1}{2}c_1^2)(\mathcal{E})[X] + \tfrac{\mathrm{rk}\,\mathcal{E}}{48} p_1(X)\right]\widehat{\upsilon} \\
&= |G|\,\widehat{\upsilon}^0 + \widehat{B}_{|G|} + |G|\,\widehat{\upsilon} \overset{(3.3),(3.6)}{=} \sqrt{|G|}\upsilon^0 = \widehat{\pi}_* \upsilon^0 \qquad (4.1) \\
&\Longrightarrow \qquad \widehat{\pi}^* \left(ch(\mathcal{E})\sqrt{\widehat{A}(X)} \right) = |G|\upsilon^0.
\end{aligned}
$$

Since it only remains to be shown that $(c_2 - \frac{1}{2}c_1^2)(\mathcal{E})[X] = 2|G|$ for binary dihedral and tetrahedral G, too, we conjecture (4.1) to hold in general. In $H^{even}(T, \mathbb{Z})$, $|G|\upsilon^0$ is the Mukai vector of a flat bundle of rank $|G|$ that naturally carries the regular representation ρ of G on the fibers, yielding a G equivariant flat bundle. Hence (4.1) is in exact agreement with the McKay correspondence. We regard this as confirmation of Theorem 3.3, though Mukai vectors do not capture any information on G equivariance, the basic ingredient of the McKay correspondence. That we need to choose the regular representation on the fiber has to do with our choices on the representative of $\widehat{B}_{|G|}$, or more precisely $\widehat{\upsilon}^0$, above. Namely, at the end of Sect. 3 we have remarked that a shift of the B-field on the underlying torus theory by an integral form induces an integral shift of $\widehat{\upsilon}^0$ by some $b \in \widehat{\Pi}_N / \oplus_s \widehat{\Gamma}_s$, that

changes the representation on our bundle of rank $|G|$ above. This freedom of choice is readily checked to correspond to the freedom of coordinate choice on the space of G equivariant flat bundles on the underlying torus T.

It is tempting to search for a direct geometric interpretation of our methods: Recall the geometric picture of Sect. 2. Here, the key point was the construction of homology classes of type III. which arise from non-generic torus cycles by removing the branch locus and then taking a single sheet of an étale covering. Similarly, the fact that $\sqrt{|G|}\upsilon^0 - B_{|G|}$ in (3.3) splits into $|G|$ lattice vectors should be interpreted such that $B_{|G|}$ corresponds to the branch locus of a $|G|$:1 branched covering of X. For $G = \mathbb{Z}_2$ such a covering exists [Nik75, (9)], and for the cyclic groups similar ones have been constructed* [Tib99], but in general not of type $|G|$:1. Moreover, we are lacking a precise mathematical formulation for the mixing of degrees in $H^{even}(X,\mathbb{Z})$ that would be needed for such an interpretation. The determination of the exact form of $B_{|G|}$ from this approach also remains under investigation, though intuitively the characterization of $B_{|G|}$ directly relates to that for the E_s in III., Sect. 2.

However, since our lattice calculations imply an interpretation in this spirit, it shall be interesting to find the appropriate mathematical framework.

Note added. After completion of this work we learned that the Kummer type lattices for cyclic groups have already been determined by J. Bertin in [Ber88]. Prop. 2.1 is in agreement with these results on Abelian orbifold constructions of $K3$.

A Lattices

The following material is taken from [Nik80a, Nik80b, Mor84].

A lattice $\Gamma \subset \mathbb{R}^{p,q}$ is called *integral*, if the associated symmetric bilinear form is an integral form. It is *even*, if the associated quadratic form is even. By $\Gamma(N)$ we denote the same $\mathbb{Z}$ module as Γ, but with quadratic form rescaled by a factor of N. The *discriminant* $\operatorname{disc}(\Gamma)$ is the determinant of the associated bilinear form on Γ. The lattice Γ is *nondegenerate* if $\operatorname{disc}(\Gamma) \neq 0$, and *unimodular* if $|\operatorname{disc}(\Gamma)| = 1$. If Γ is a nondegenerate integral lattice, then $\operatorname{disc}(\Gamma) = |\Gamma^*\colon \Gamma|$, where Γ^* denotes the dual lattice of Γ and $\Gamma \hookrightarrow \Gamma^*$ by using the bilinear form on Γ. The *signature* (p,q) of Γ is the multiplicity of the eigenvalues $(+1,-1)$ for the induced quadratic form on $\Gamma \otimes \mathbb{R}$. The

*I thank Claus Hertling for his explanations on this point and for prodding me to the relevant literature.

discriminant form q_Γ associated to an even lattice Γ is the map $q_\Gamma : \Gamma^*/\Gamma \to \mathbb{Q}/2\mathbb{Z}$ which is induced by the quadratic form of Γ, together with the induced symmetric bilinear form on Γ^*/Γ with values in $\mathbb{Q}/\mathbb{Z}$.

The examples of even unimodular lattices most frequently used in our work are the *hyperbolic lattice* U with quadratic form given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the lattice E_8 with quadratic form given by the Cartan matrix of E_8. Moreover, one has

Theorem A.1 [Mil58]
If Γ is an even unimodular lattice with signature $(p, p+\delta)$, $p > 0$, $\delta \geq 0$, then

$$\delta \equiv 0(8), \quad \Gamma \cong \Gamma^{p,p+\delta} := U^p \oplus (E_8(-1))^{\delta/8}.$$

A sublattice $\Lambda \subset \Gamma$ is *primitive* iff Γ/Λ is free. A vector $\lambda \in \Gamma$ is primitive, if $\Lambda := \mathbb{Z}\lambda \subset \Gamma$ is primitive.

Embeddings of primitive sublattices in unimodular lattices are characterized by

Theorem A.2 [Nik80a, Prop. 1.6.1], [Nik80b, §1]
Let Λ denote a primitive nondegenerate sublattice of an even unimodular lattice Γ. Then the embedding $\Lambda \hookrightarrow \Gamma$ with $\Lambda^\perp \cap \Gamma \cong \mathcal{V}$ is specified by an isomorphism $\gamma : \Lambda^*/\Lambda \to \mathcal{V}^*/\mathcal{V}$, such that for the discriminant forms $q_\Lambda = -q_\mathcal{V} \circ \gamma$. Moreover,

$$\Gamma \cong \left\{ (\lambda, v) \in \Lambda^* \oplus \mathcal{V}^* \mid \gamma(\overline{\lambda}) = \overline{v} \right\},$$

where $\overline{l}$ denotes the projection of $l \in L^*$ onto L^*/L.

B Grassmannians

By $\mathcal{T}^{a,b}$ we denote the *Grassmannian of oriented positive definite subspaces* $W \subset \mathbb{R}^{a,b}$ *with* $\dim W = a$. Hence

$$\mathcal{T}^{a,b} \cong O^+(a,b)/SO(a) \times O(b),$$

where $O(a,b) = O(\mathbb{R}^{a,b})$ and analogously for $O^+(a,b)$, $SO(a,b)$, $O(a,b)$, and $O(a) = O(a,0)$ etc. Here, for any vector space W with scalar product, $O(W)$ denotes the group of orthogonal transformations of W. Its subgroup $O^+(W)$ contains all elements that do not interchange the two components of the space of maximal positive definite subspaces of W. Note that for positive definite W, $SO(W) = O^+(W)$. For a lattice $\Gamma \subset W$ the group $O(\Gamma)$ is the group of lattice automorphisms of Γ, and $O^+(\Gamma) = O(\Gamma) \cap O^+(W)$ etc.

With the techniques of [BS73] one shows:

Theorem B.1
For $a, b \in \mathbb{N}$, there is an isomorphism

$$\mathcal{T}^{a+1,b+1} \cong \mathcal{T}^{a,b} \times \mathbb{R}^+ \times \mathbb{R}^{a,b},$$

which is specified by the choice of two null vectors $\upsilon, \upsilon^0 \in \mathbb{R}^{a+1,b+1}$ with $\langle \upsilon, \upsilon^0 \rangle = 1$ such that $\mathbb{R}^{a,b} \perp \upsilon, \upsilon^0$, and $\mathcal{T}^{a,b}$ is built on $\mathbb{R}^{a,b}$ in the above product. Explicitly, we have

$$\begin{array}{rcl} x \longmapsto (\Sigma, V, B) \quad \Longleftrightarrow \quad x & = & \mathrm{span}_{\mathbb{R}} \left(\xi(\Sigma),\, \upsilon^0 + B + (V - B^2/2)\upsilon \right), \\ \xi(\sigma) & = & \sigma - \langle B, \sigma \rangle \upsilon. \end{array} \tag{B.1}$$

The above isomorphism induces the structure of a warped product on $\mathcal{T}^{a,b} \times \mathbb{R}^+ \times \mathbb{R}^{a,b}$.

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